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Elastic beam in adhesive contact

W. Han [a,1], K.L. Kuttler [b], M. Shillor [c,*], M. Sofonea [d]

[a] *Department of Mathematics, University of Iowa, Iowa City, IA 52242, USA*
[b] *Department of Mathematics, Brigham Young University Provo, UT 84602, USA*
[c] *Department of Mathematics and Statistics, Oakland University, Rochester, MI 48309, USA*
[d] *Laboratorie de Théorie des Systèmes, Université de Perpignan 66 860 Perpignan, France*

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Abstract

Dynamic and quasistatic processes of contact with adhesion between an elastic or viscoelastic beam and a foundation are considered. The contact is modeled with the Signorini condition when the foundation is rigid, and with normal compliance when it is deformable. The adhesion is modeled by introducing the bonding function β, the evolution of which is described by an ordinary differential equation. The existence and uniqueness of the weak solution for each of the problems is established using the theory of variational inequalities, fixed point arguments and the existence and uniqueness result in Commun. Contemp. Math. 1(1) (1999) 87–123. The numerical approximations of the quasistatic problem with normal compliance are considered, based on semi-discrete and fully discrete schemes. The convergence of the solutions of the discretized schemes is proved and error estimates for these approximate solutions are derived.

Keywords: Dynamic or quasistatic contact; Beam; Signorini condition; Normal compliance; Adhesion; Weak solutions; Semi-discrete and fully-discrete schemes; Error estimates

1. Introduction

Processes of adhesion are very important in industry, especially when composite materials are involved. There exists extensive engineering literature on various aspects of the subject. However, general mathematically sound models are very recent. A novel approach to the modeling of contact with adhesion, based on thermodynamic derivation, can be found in Frémond (1982, 1987). There, the adhesive contact process has been modeled by the introduction of an internal variable, the adhesion field β, that measures the fraction of active bonds.

Recent modeling, analysis, and numerical simulations of adhesive contact with or without friction can be found in Chau et al. (2001, in preparation), Raous et al. (1999) and references therein. The static contact problem for the elastoplastic beam can be found in Khludnev and Hoffmann (1992). In Chau et al. (2001)

* Corresponding author. Tel.: +1-248-3703430; fax: +1-248-3704184.
E-mail address: shillor@oakland.edu (M. Shillor).

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the dynamic frictionless adhesive contact problem has been modeled and analyzed, and the quasistatic version has been considered in Chau et al. (in preparation) where numerical simulations have been presented. The problem of adhesive contact with friction can be found in Raous et al. (1999) where the modeling, based on thermodynamic principles, can be found, and numerical analysis and simulations presented.

This work deals with the adhesive contact between a beam and a foundation. The beam is assumed to be either elastic or viscoelastic, and the obstacle either rigid or deformable. Our interest lies in the description and analysis of the dynamic or quasistatic processes of contact when adhesion takes place during contact. We also describe convergent semi-discrete and fully discrete numerical schemes for the quasistatic problem with deformable foundation. We note that very recent results on quasistatic contact of an elastic body can be found in Andersson (1999, 2000). In the first paper the normal compliance contact condition is employed, and in the second paper the author obtained the Signorini condition by passing to the normal compliance limit.

The paper is organized as follows: In Section 2, we describe the classical model for the process of adhesive quasistatic contact between an elastic beam and a rigid foundation. In Section 3, we present the variational formulation of the problem, list the assumptions on the problem data, and state an existence and uniqueness result in Theorem 3.1. The proof of the Theorem is given in Section 4, and is based on the theory of time-dependent variational inequalities and the Banach fixed point theorem. In Section 5, we present dynamic and quasistatic models for adhesive contact between a viscoelastic or an elastic beam and a deformable foundation. The reaction force is modeled with the 'normal compliance' condition. We prove the existence of the unique solution to each of the problems. Moreover, we show that in the quasistatic case when the foundation becomes stiffer the solutions approach the solution of the problem with a rigid foundation. In Section 6 we consider the spatially semi-discrete approximation of the quasistatic problem with normal compliance. We show that the discretized approximations converge to the solution. Under additional regularity assumption we also establish the rate of convergence. Finally, in Section 7, we obtain similar results for the fully discrete problem.

2. The model

In this section, we construct a model for the quasistatic contact process with adhesion. We consider a linearly elastic beam of length L that is clamped at its left end while the right end is free. The beam is being acted upon by an applied force of (linear) density f, and it may come in adhesive contact with a rigid foundation below it. The setting is depicted in Fig. 1. The cases of a viscoelastic beam, dynamic process or a deformable foundation will be described in Section 5.

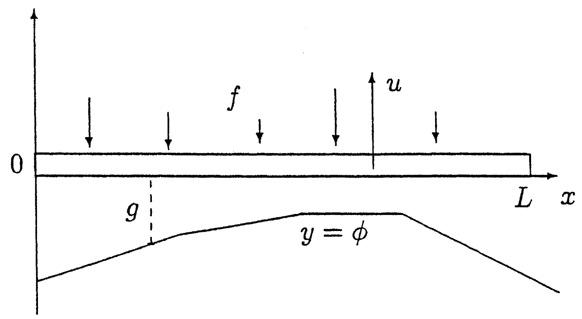

Fig. 1. The setting of the problem.

We denote $\Omega_T = (0, L) \times (0, T)$, for $T > 0$, and let $u = u(x,t)$ represent the vertical displacement of the beam, at $(x,t) \in \bar{\Omega}_T$. We assume that the obstacle is described by the function $y = \phi(x)$, for $0 \leqslant x \leqslant L$. We denote $A = EI$, where I is the beam's moment of inertia and E the Young modulus, and let $L_A(u)$ be the function

$$L_A(u) = \frac{\partial^2}{\partial x^2}\left(A\frac{\partial^2 u}{\partial x^2}\right). \tag{2.1}$$

We assume, first, that the acting forces vary slowly in time and, therefore, the process is quasistatic. Then, the equation of motion for the beam is

$$L_A(u) = f + \xi \quad \text{in } \Omega_T, \tag{2.2}$$

where $\xi = \xi(x,t)$ denotes the reaction force of the foundation and the adhesion force. The rigid foundation restricts the motion of the beam to displacements above it, thus

$$u \geqslant \phi \quad \text{in } \Omega_T, \tag{2.3}$$

which represents a non-penetration condition. When contact takes place the foundation's reaction force ξ is directed upward,

$$u = \phi \Rightarrow \xi \geqslant 0 \quad \text{in } \Omega_T. \tag{2.4}$$

We now describe the adhesion process, following Frémond (1982, 1987). We introduce the internal state variable $\beta = \beta(x,t)$, the 'bonding field', which measures the fraction of the active bonds between the beam and the foundation. When $\beta = 1$ at a point the adhesion is complete; $\beta = 0$ means that all the bonds are severed and there is no adhesion, and $0 < \beta < 1$ represents the state of partial bonding. We suppose that the adhesive resistance is active when the force is directed upwards, trying to separate the beam from the foundation, and this restoring force is proportional to the distance from the obstacle and to β^2. Therefore,

$$u > \phi \Rightarrow \xi = -\kappa(u - \phi)\beta^2 \quad \text{in } \Omega_T, \tag{2.5}$$

(see also Raous et al., 1999). Here, $\kappa > 0$ represents the interface stiffness when the adhesion is complete, and $\kappa\beta^2$ is the 'spring constant' of the bonding field.

Now, conditions (2.3)–(2.5) may be written in the following complementary form:

$$u \geqslant \phi, \quad \xi + \kappa(u - \phi)\beta^2 \geqslant 0, \quad (u - \phi)(\xi + \kappa(u - \phi)\beta^2) = 0 \quad \text{in } \Omega_T. \tag{2.6}$$

Next, following Raous et al. (1999) we assume that the evolution of the adhesion field is given by

$$\beta' = -\gamma\kappa(u - \phi)^2(\beta)_+ \quad \text{in } \Omega_T, \tag{2.7}$$

where γ is the adhesion rate, assumed to be a positive constant, and $r_+ = \max\{r, 0\}$ denotes the positive part of r. We use the latter to ensure that β does not become negative. In Eq. (2.7), and everywhere in the sequel, a prime represents the time derivative.

We note that in Eq. (2.7) once debonding takes place there is no rebonding, i.e., $\beta' \leqslant 0$. If we deal with a process where rebonding can happen, condition (2.7) had to be modified accordingly (see e.g., Chau et al. (2001)).

To complete the model we prescribe appropriate initial and boundary conditions. The initial condition takes the from

$$\beta(x, 0) = \beta_0(x) \quad \text{for } x \in (0, L), \tag{2.8}$$

where β_0 represents the initial bonding field. The beam is rigidly attached as its left end, thus,

$$u(0,t) = u_x(0,t) = 0 \quad \text{for } t \in [0, T]. \tag{2.9}$$

In this work, subscripts x, xx, and xxx denote the first, second and third partial derivatives with respect to x, respectively. There are no moments acting on the free end of the beam, thus,

$$u_{xx}(L,t) = u_{xxx}(L,t) = 0 \quad \text{for } t \in [0,T]. \tag{2.10}$$

The classical statement of the problem of quasistatic adhesive contact of a beam with a rigid obstacle is:

Problem P. Find a displacement function $u : \bar{\Omega}_T \to \mathrm{R}$ and an adhesion function $\beta : \bar{\Omega}_T \to \mathrm{R}$ such that Eqs. (2.2), (2.6)–(2.10) hold.

3. Variational formulation and statement of results

We deal with a contact or obstacle problem, and it is well known that there exists a regularity ceiling for the solutions which, generally, prevents them from having all the classical derivatives needed for the classical formulation to make sense. Therefore, we proceed to derive a weak or variational formulation of the problem.

First, we introduce additional notation. We use standard notation for L^p and Sobolev spaces (see e.g., Adams, 1975; Ionescu and Sofonea, 1993; Lions and Magenes, 1972) and let V be the closed subspace of $H^2(0,L)$ given by

$$V = \{v \in H^2(0,L) | v(0) = v_x(0) = 0\}.$$

We denote by H the space $L^2(0,L)$ and by $(\cdot,\cdot)_H, |\cdot|_H$ its inner product and the associate norm, respectively. Let K denote the convex subset of V defined by

$$K = \{v \in V | v \geqslant \phi \text{ on } [0,L]\}.$$

If $(X, |\cdot|_X)$ is a real normed space, we denote by $C(0,T;X)$ and $C^1(0,T;X)$ the space of continuous and continuously differentiable functions from $[0,T]$ to X, with the respective norms

$$|u|_{C(0,T;X)} = \max_{t\in[0,T]} |u(t)|_X, \quad |u|_{C^1(0,T;V)} = \max_{t\in[0,T]} |u(t)|_X + \max_{t\in[0,T]} |u'(t)|_X.$$

In the study of problem P we assume the following on the data:

$$A \in L^\infty(0,L) \text{ and there exists } A_0 > 0 \text{ such that } A \geqslant A_0 \text{ a.e. on } (0,L), \tag{3.1}$$

$$f \in C(0,T;L^2(0,L)), \tag{3.2}$$

$$\gamma = \text{constant} > 0, \tag{3.3}$$

$$\phi \in C^1(0,L), \quad \phi(0) \leqslant 0, \text{ and if } \phi(0) = 0 \text{ then } \phi'(0) \leqslant 0, \tag{3.4}$$

$$\kappa \in L^\infty(0,L), \quad \kappa \geqslant 0 \text{ a.e. on } (0,L), \tag{3.5}$$

$$\beta_0 \in L^\infty(0,L), \quad 0 < \beta_0 \leqslant 1 \text{ a.e. on } (0,L). \tag{3.6}$$

For the sake of simplicity we choose γ to be a constant, our results hold true for the case when $\gamma \in L^\infty(0,L)$ and $\gamma \geqslant \gamma_* > 0$, for some constant γ_*. We note that condition (3.4) guarantees that the set K is not empty.

Let $a : V \times V \to \mathrm{R}$ be the functional

$$a(u,v) = \int_0^L A u_{xx} v_{xx} \,\mathrm{d}x \quad \forall u,v \in V. \tag{3.7}$$

We now obtain a variational formulation of the mechanical problem P. To that end we assume that u, ξ, and β are smooth functions satisfying Eqs. (2.2), (2.6)–(2.9). Let $v \in K$ be a test function and let $t \in [0, T]$. We multiply Eq. (2.2) by $v - u(t)$, thus

$$\int_0^L L_A(u(t))(v - u(t))\,\mathrm{d}x = \int_0^L f(t)(v - u(t))\,\mathrm{d}x + \int_0^L \xi(t)(v - u(t))\,\mathrm{d}x.$$

Using now Eq. (2.1), performing two integrations by parts and keeping in mind Eqs. (2.9) and (2.10) we obtain

$$\int_0^L L_A(u(t))(v - u(t))\,\mathrm{d}x = \int_0^L Au_{xx}(t)(v_{xx} - u_{xx}(t))\,\mathrm{d}x.$$

Moreover, from Eq. (2.6) we deduce that

$$\begin{aligned}\xi(v-u) &= (\xi + \kappa(u-\phi)\beta^2)(v-u) - \kappa(u-\phi)\beta^2(v-u)\\ &= (\xi + \kappa(u-\phi)\beta^2)(v-\phi) + (\xi + \kappa(u-\phi)\beta^2)(\phi - u) - \kappa(u-\phi)\beta^2(v-u)\\ &\geqslant -\kappa(u-\phi)\beta^2(v-u),\end{aligned}$$

and, therefore,

$$\int_0^L \xi(t)(v - u(t))\,\mathrm{d}x \geqslant \int_0^L \kappa\beta^2(u(t) - \phi)(u(t) - v)\,\mathrm{d}x. \tag{3.8}$$

Thus, we obtain

$$a(u(t), v - u(t)) + (\kappa\beta^2(t)(u(t) - \phi), v - u(t))_H \geqslant (f(t), v - u(t))_H, \tag{3.9}$$

for $0 \leqslant t \leqslant T$. Since Eqs. (2.6) and (2.9) imply that $u(t) \in K$, Eqs. (2.7), (2.8) and (3.9) yield the following variational formulation of Problem P.

Problem P_V. Find a displacement function $u : [0, T] \to V$ and an adhesion function $\beta : [0, T] \to L^\infty(0, L)$ such that, for all $t \in [0, T]$,

$$u(t) \in K, \tag{3.10}$$

$$a(u(t), v - u(t)) + (\kappa\beta^2(t)(u(t) - \phi), v - u(t))_H \geqslant (f(t), v - u(t))_H \quad \forall\, v \in K, \tag{3.11}$$

$$\beta'(t) + \gamma\kappa(u(t) - \phi)^2(\beta(t))_+ = 0 \quad \text{in } \Omega_T, \tag{3.12}$$

$$\beta(0) = \beta_0 \quad \text{a.e. on } (0, L). \tag{3.13}$$

Our main result, which we establish in the next section, is the following:

Theorem 3.1. *Assume that conditions* (3.1)–(3.6) *hold. Then, there exists a unique solution* $\{u, \beta\}$ *of Problem* P_V. *Moreover, the solution satisfies*

$$u \in C(0, T; V), \quad \beta \in C^1(0, T; L^\infty(0, L)). \tag{3.14}$$

We conclude that, under the assumptions (3.1)–(3.6), the mechanical problem (2.2), (2.6)–(2.10) has a unique weak solution $\{u, \beta\}$.

4. Proof of Theorem 3.1

The proof of the theorem will be carried out in several steps. It is based on time-dependent variational inequalities and a fixed point theorem. Everywhere in this section C will represent a positive generic constant which is independent of t and β and whose value may change from line to line. For the sake of simplicity, we assume below that $\phi \equiv 0$, otherwise, one needs to replace $u(t)$ with $(u(t) - \phi)$ in the appropriate places below.

We start by defining an appropriate inner product on the space V. To this end we observe that there exists $C > 0$ such that $C|v|_H \leqslant |v_x|_{L^2(0,L)}$ for all $v \in H^1(0,L)$ satisfying $v(0) = 0$, thus,

$$C|v|_{H^2(0,L)} \leqslant |v_{xx}|_H \quad \forall v \in V. \tag{4.1}$$

We consider now the inner product on V given by

$$(u,v)_V = (u_{xx}, v_{xx})_H, \tag{4.2}$$

and let $|.|_V$ be the associated norm. By using Eq. (4.1) we find that $|.|_{H^2(0,L)}$ and $|.|_V$ are equivalent norms on V and, therefore, $(V, (.,.)_V)$ is a real Hilbert space.

The first step of the proof is the following simple lemma.

Lemma 4.1. *Let Eqs.* (3.1), (3.2) *and* (3.5) *hold. If* $\beta \in C(0,T;L^\infty(0,L))$ *is given, then there exists a unique solution* $u \in C(0,T;V)$ *which satisfies Eqs.* (3.10) *and* (3.11), *for all* $t \in [0,T]$. *Moreover,*

$$|u(t)|V \leqslant C, \tag{4.3}$$

where C is independent of β.

Proof. Using Eqs. (3.1), (3.7), and (4.2) we find that a is a bilinear continuous and coercive form on V, that is

$$|a(u,v)| \leqslant C|u|_V|v|_V \quad \forall u, v \in V, \tag{4.4}$$

$$a(v,v) \geqslant C|v|_V^2 \quad \forall v \in V. \tag{4.5}$$

For all $t \in [0,T]$, let $B(t) : V \to V'$ be the operator

$$(B(t)u, v)_V = a(u,v) + (\kappa\beta^2(t)u, v)_H \quad \forall u, v \in V.$$

Using Eqs. (3.5), (4.4), and (4.5) it follows that $B(t)$ is a strongly monotone Lipschitz continuous operator on V, and K is a nonempty closed convex set of V. It follows from standard results (see e.g., Brezis, 1968; Duvaut and Lions, 1976; Kinderlehrer and Stampacchia, 1980 or Lions, 1969), that for each $t \in [0,T]$ there exists a unique element $u(t) \in V$ which solves Eqs. (3.10) and (3.11). Choosing $v = 0$ in Eq. (3.11) and using Eq. (4.5) we obtain Eq. (4.3).

Now, let t_1, $t_2 \in [0,T]$ and for the sake of simplicity we denote $u(t_i) = u_i$, $\beta(t_i) = \beta_i$, $f(t_i) = f_i$. Using Eqs. (3.10), (3.11) and algebraic manipulations we find

$$a(u_1 - u_2, u_1 - u_2) + (k\beta_1^2 u_1 - k\beta_2^2 u_2, u_1 - u_2)_H \leqslant (f_1 - f_2, u_1 - u_2)_H. \tag{4.6}$$

Now, from Eqs. (4.3), (4.5), and (4.6) we find

$$C|u_1 - u_2|_V \leqslant |f_1 - f_2|_H + |\beta_1 - \beta_2|_{L^\infty(0,L)}|\beta_1 + \beta_2|_{L^\infty(0,L)}. \tag{4.7}$$

Since $\beta \in C(0,T;L^\infty(0,L))$ we obtain from Eqs. (3.2) and (4.7) that $u \in C(0,T;V)$, which concludes the proof. $\square$

Let u_β denote the solution in Lemma 4.1, and consider the initial value problem

$$\theta' + \gamma\kappa u_\beta^2(\theta)_+ = 0 \quad \text{in } \Omega_T, \tag{4.8}$$

$$\theta(0) = \beta_0 \quad \text{a.e. on } (0, L). \tag{4.9}$$

Clearly, under the assumptions of Theorem 3.1, there exists a unique function $\theta = \theta(\beta) \in C^1(0, T; L^\infty(0, L))$ which solves Eqs. (4.8) and (4.9). Let Z denote the closed subset of $C(0, T; L^\infty(0, L))$ which is defined as

$$Z = \{\beta \in C(0, T; L^\infty(0, L)) \,|\, \beta(x, t) \in [0, 1], \text{ a.e. } x \in (0, L), \text{ for all } t \in [0, T]\}, \tag{4.10}$$

and assume that Eqs. (3.1)–(3.6) hold. Then,

Lemma 4.2. *If $\beta \in Z$ then $\theta(\beta) \in Z$.*

Proof. The result follows from Eq. (4.8) and the assumption that $\beta_0(x) \in (0, 1]$ for a.e. $x \in (0, L)$. Indeed, Eq. (4.8) implies that for a.e. $x \in (0, L)$, the function $t \mapsto \theta(\beta)(x, t)$ is decreasing and its derivative vanishes when $\kappa u_\beta^2 \theta(\beta)(x, t) \leqslant 0$, implying that $\theta(\beta)(x, t) \geqslant 0$ a.e. on Ω_T. □

We now have all the ingredients to prove Theorem 3.1. Suppose β_i, $i = 1, 2$, are two functions in Z and let $t \in [0, T]$. We need to compare the functions $u_1 = u_{\beta_1}$ and $u_2 = u_{\beta_2}$.

Since $\beta_1, \beta_2 \in Z$, by using arguments similar to those used in the proof of Eq. (4.7), we find

$$|u_1(t) - u_2(t)|_V \leqslant C|\beta_1(t) - \beta_2(t)|_{L^\infty(0,L)}.$$

This implies, by the continuity of the embedding of V into $L^\infty(0, L)$, that

$$|u_1(t) - u_2(t)|_{L^\infty(0,L)} \leqslant C|\beta_1(t) - \beta_2(t)|_{L^\infty(0,L)}. \tag{4.11}$$

Now, Eqs. (4.3), (4.8), (4.9) and the continuity of the embedding of V into $L^\infty(0, L)$, yield

$$\begin{aligned} |\theta(\beta_1)(t) - \theta(\beta_2)(t)|_{L^\infty(0,L)} &\leqslant \int_0^t |\gamma\kappa u_1^2(s)\theta(\beta_1)(s) - \gamma\kappa u_2^2(s)\theta(\beta_2)(s)|_{L^\infty(0,L)}\,\mathrm{d}s \\ &\leqslant C\int_0^t |u_1(s) - u_2(s)|_{L^\infty(0,L)}\,\mathrm{d}s + C\int_0^t |\theta(\beta_1)(s) - \theta(\beta_2)(s)|_{L^\infty(0,L)}\,\mathrm{d}s. \end{aligned}$$

Using a Gronwall-type inequality we obtain

$$|\theta(\beta_1)(t) - \theta(\beta_2)(t)|_{L^\infty(0,L)} \leqslant C\int_0^t |u_1(s) - u_2(s)|_{L^\infty(0,L)}\,\mathrm{d}s. \tag{4.12}$$

Thus, Eqs. (4.11) and (4.12) yield

$$|\theta(\beta_1)(t) - \theta(\beta_2)(t)|_{L^\infty(0,L)} \leqslant C\int_0^t |\beta_1(s) - \beta_2(s)|_{L^\infty(0,L)}\,\mathrm{d}s. \tag{4.13}$$

Iterating this inequality n times, we deduce

$$|\theta^n(\beta_1) - \theta^n(\beta_2)|_{C(0,T;L^\infty(0,L))} \leqslant \frac{C^n T^n}{n!}|\beta_1 - \beta_2|_{C(0,T;L^\infty(0,L))}.$$

Therefore, θ^n is a contraction mapping on Z, for all n sufficiently large, and hence θ has a unique fixed point in Z which is the unique solution of Theorem 3.1.

5. Models with deformable support

In this section we describe and analyze several versions of the model in which the support is deformable. We use the so-called 'normal compliance' condition to describe the reaction of the foundation when $u < \phi$. We consider the dynamic problem with and without viscosity along with the quasistatic problem. We establish existence and uniqueness results for these problems. We also consider the manner in which the solutions of the quasistatic problems with normal compliance converge to those of the quasistatic problem in which the support is rigid. For the sake of simplicity, and without loss of generality, we assume that $\phi \equiv 0$. All the results below apply to the case when ϕ satisfies Eq. (3.4).

We use the following well-known theorem found in Lions (1969).

Theorem 5.1. *Let $p \geqslant 1$, $q > 1$ and let $W \subseteq U \subseteq Y$ be Banach spaces with compact inclusion map $i : W \to U$ and continuous inclusion map $i : U \to Y$. Then, the set*

$$S_R = \{\boldsymbol{u} \in L^p(0,T;W) | \mathbf{u}' \in L^q(0,T;Y), \quad \|\mathbf{u}\|_{L^p(0,T;W)} + \|\mathbf{u}'\|_{L^q(0,T;Y)} < R\}$$

is precompact in $L^p(0,T;U)$.

Also, we will use the following theorem found in Seidman (1989) and Simon (1987).

Theorem 5.2. *Let $q > 1$ and let W, U and Y be as in Theorem 5.1. Then, the set*

$$S_{RT} = \{\mathbf{u} | \|\mathbf{u}(t)\|_W + \|\mathbf{u}'\|_{L^q(0,T;Y)} \leqslant R, t \in [0,T]\}$$

is precompact in $C(0,T;U)$.

We consider the same setting as in Section 2, but the foundation now is deformable and its reaction force f_{R} depends on the beams displacement. We use a 'normal compliance' condition to describe it (see, e.g., Kikuchi and Oden, 1988; Klarbring et al., 1988), thus,

$$f_{\mathrm{R}} = p(u). \tag{5.1}$$

Here, $p = p(\cdot) \geqslant 0$ is a given decreasing, globally Lipschitz continuous function which vanishes for non-negative values of its argument, since when $u > 0$ there is loss of contact. We consider the case where $p(\cdot)$ is Lipschitz for the sake of simplicity, and more general functions are possible (cf. Kuttler and Shillor, 1999).

The total force acting on the beam consists of the external force f, the adhesive force ξ and the foundation reaction f_{R}. Thus, the dynamic equation of the beam in the elastic case is

$$v' + L_A(u) = f + \xi + p(u) \quad \text{in } \Omega_T, \tag{5.2}$$

where v is the velocity, $v = u'$. To include the effects of viscosity, let B be a function satisfying $B \in L^\infty(0,L)$, and there exists $B_0 > 0$ such that $B \geqslant B_0$ a.e. on $(0,L)$. Now, let $L_B(v)$ be defined similarly to $L_A(u)$, representing internal viscosity of the Kelvin–Voigt type, and adding this term in Eq. (5.2) gives

$$v' + L_B(v) + L_A(u) = f + \xi + p(u) \quad \text{in } \Omega_T. \tag{5.3}$$

The associated quasistatic elastic equation is

$$L_A(u) = f + \xi + p(u) \quad \text{in } \Omega_T. \tag{5.4}$$

We assume that the adhesion is described by Eq. (2.5), but the unilateral condition (2.6) does not hold anymore, since we allow for $u < 0$. Next, the evolution equation for the adhesion field (2.7) is modified as follows:

$$\beta' + \gamma\kappa((u)_+)^2(\beta)_+ = 0 \quad \text{in } \Omega_T. \tag{5.5}$$

We use $(u)_+$ to ensure that only tension contributes to debonding, since compression does not affect it. Clearly, a more general condition can be employed, such as allowing for rebonding (cf. Chau et al., 2001), but we will not pursue it here. We also assume that the boundary conditions are given by Eqs. (2.9) and (2.10).

Next, we define the operators, $\bar{A}$, $\bar{B}$, P, and $S(\beta,\cdot)$ as follows:

$$\langle \bar{A}u, w\rangle = \int_0^L A u_{xx} w_{xx}\,\mathrm{d}x, \tag{5.6}$$

$$\langle \bar{B}v, w\rangle = \int_0^L B v_{xx} w_{xx}\,\mathrm{d}x, \tag{5.7}$$

$$\langle P(u), w\rangle = -\int_0^L p(u) w\,\mathrm{d}x, \tag{5.8}$$

$$\langle S(\beta,u), w\rangle = \int_0^L \kappa\beta^2 u w\,\mathrm{d}x. \tag{5.9}$$

Let $\mathscr{V} \equiv L^2(0,T;V)$, $\mathscr{V}' \equiv L^2(0,T;V')$, and denote $X \equiv \{v \in \mathscr{V} : v' \in \mathscr{V}'\}$. Proceeding as in Section 3, we obtain from Eq. (5.5) and the boundary conditions the following abstract formulation of our problem: Find $v \in \mathscr{V}$ and $\beta \in C(0,T;L^\infty(0,L))$, such that

$$(\eta v)' + \eta\bar{B}v + \bar{A}u + S(\beta,u) + P(u) = f \quad \text{in } \mathscr{V}', \tag{5.10}$$

$$\beta' + \gamma\kappa((u)+)^2(\beta)_+ = 0, \quad \beta' \in C(0,T;L^\infty(0,L)), \tag{5.11}$$

$$\beta(0) = \beta_0, \;\; \eta v(0) = \eta v_0, \tag{5.12}$$

and

$$u(t) = u_0 + \int_0^t v(s)\,\mathrm{d}s. \tag{5.13}$$

Here, $\eta = 0, 1$; when $\eta = 1$ the problem is dynamic with viscosity, and when $\eta = 0$ we have the quasistatic inviscid problem. We assume that

$$\beta_0(x) \in (0,1], \quad v_0 \in H, \;\; u_0 \in V.$$

Theorem 5.3. *Under the above assumptions there exists a unique solution of problem* (5.10)–(5.13), *for* $\eta =$ 0, 1.

Proof. In the case $\eta = 0$ existence and uniqueness follow from similar considerations as in Section 4. Therefore, we consider the case when $\eta = 1$. The proof of existence and uniqueness may be carried out along the lines presented in Section 4, too. Suppose that $\beta \in C(0,T;L^\infty(0,L))$ with $\beta(t) \in [0,1]$ for a.e. $x \in (0,L)$, is given. We consider Eq. (5.10) with given β, and note the following. The operators, $\bar{A}$, $\bar{B} : \mathscr{V} \to \mathscr{V}'$ are monotone and linear. Indeed, for $D = \bar{A}$, $\bar{B}$, $\langle Du_1 - Du_2, u_1 - u_2\rangle \geqslant C|u_1 - u_2|_V^2$. Also,

$$\int_0^t \langle \bar{A}u_1 - \bar{A}u_2, v_1 - v_2 \rangle \, \mathrm{d}s = \int_0^t (A(u_{1xx} - u_{2xx}), (v_{1xx} - v_{2xx}))_H \, \mathrm{d}s$$
$$= \frac{1}{2} \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} (A(u_{1xx} - u_{2xx}), (u_{1xx} - u_{2xx})) \, \mathrm{d}s$$
$$= \frac{1}{2} (A(u_{1xx}(t) - u_{2xx}(t)), (u_{1xx}(t) - u_{2xx}(t)))_H \geqslant 0.$$

The operators $S(\beta, \cdot)$ and P are completely continuous as mappings from X to X' by Theorem 5.1. Therefore, the operator, $v \to Bv + Au + S(\beta, u) + P(u)$ is pseudomonotone as a map from X to X', and is bounded as a map from $\mathscr{V}$ to $\mathscr{V}'$. Moreover,

$$\langle \bar{B}v + \bar{A}u + S(\beta, u) + P(u), v \rangle_{\mathscr{V}', \mathscr{V}} \geqslant B_0 \|v\|_{\mathscr{V}}^2 - C(u_0) \|v\|_{\mathscr{V}},$$

which implies that the operator is coercive. Therefore, we can apply the existence theorem of Kuttler and Shillor (1999) and conclude that there exists a solution of the abstract problem (5.10) and (5.13). Next, suppose v_i, for $i = 1, 2$, are two solutions of Eq. (5.10) with given β. Then, it follows from Eq. (5.10) that

$$\frac{1}{2} |v_1(t) - v_2(t)|_H^2 + B_0 \int_0^t \|v_1 - v_2\|_V^2 \, \mathrm{d}s + A_0 \|u_1(t) - u_2(t)\|_V^2$$
$$+ \int_0^t (\kappa \beta^2(s)(u_1(s) - u_2(s)), v_1(s) - v_2(s))_H \, \mathrm{d}s \leqslant C \left(\int_0^t |v_1(s) - v_2(s)|_H \, \mathrm{d}s \right)^2.$$

Now, using the boundedness of β, Jensen's inequality, and the compactness of the embedding of V into H, we obtain

$$|v_1(t) - v_2(t)|_H^2 + \int_0^t \|v_1 - v_2\|_V^2 \, \mathrm{d}s + \|u_1(t) - u_2(t)\|_V^2 \leqslant C \int_0^t |v_1(s) - v_2(s)|_H^2 \, \mathrm{d}s,$$

which implies that $v_1 = v_2$ by Gronwall's inequality. Here, C is independent of β, for β as above having values in $[0, 1]$.

Multiplying Eq. (5.10) by v and integrating from 0 to t, we find, after routine manipulations, that

$$\frac{1}{2} \left(|v(t)|_H^2 - |v_0|_H^2 \right) + B_0 \int_0^t \|v\|_V^2 \, \mathrm{d}s + A_0 \left(\|u(t)\|_V^2 - \|u_0\|_V^2 \right) + \int_0^L \Phi(u(x, t)) \, \mathrm{d}x \leqslant \int_0^t |f| |v| \, \mathrm{d}s,$$

where $\Phi'(r) = -p(r)$ and $\Phi(r) \geqslant 0$. It follows from Gronwall's inequality that there exists a constant C, independent of β and B_0, such that

$$|v(t)|_H^2 + B_0 \int_0^t \|v\|_V^2 \, \mathrm{d}s + A_0 \|u(t)\|_V^2 \leqslant C. \tag{5.14}$$

As in Section 4, we define a mapping $\Theta : Z \to Z$ for Z (Z given in Eq. (4.10)) as follows: For $\beta \in Z$, let v_β and u_β be the solution of Eq. (5.10) with the initial data in Eq. (5.12). Then $\Theta(\beta) \in Z$ is the solution of Eq. (5.11) with the initial data in Eq. (5.13) and u is replaced with u_β. We need to consider how does Θ depend on β. Let β_i, $i = 1, 2$ be two elements of Z, and denote by v_i and u_i the variables, v_{β_i} and u_{β_i}, respectively. There is only one term in Eq. (5.10) containing β and from this term, along with the assumption that $\beta(x, t) \in [0, 1]$, we obtain

$$
\begin{aligned}
|\langle S(\beta_1,u_1)-S(\beta_2,u_2),v_1-v_2\rangle| &\leqslant \left|\int_0^L \kappa\beta_1^2(u_1-u_2)(v_1-v_2)\,\mathrm{d}x\right| \\
&+\left|\int_0^L \kappa(\beta_1^2-\beta_2^2)u_2(v_1-v_2)\,\mathrm{d}x\right| \leqslant \kappa|u_1-u_2|_H|v_1-v_2|_H \\
&+2\kappa\int_0^L|\beta_1-\beta_2||u_2||v_1-v_2|\,\mathrm{d}x \leqslant \kappa|u_1-u_2|_H|v_1-v_2|_H+2\kappa C\int_0^L|\beta_1-\beta_2||v_1-v_2|\,\mathrm{d}x,
\end{aligned}
\tag{5.15}
$$

thanks to the inequality (5.14) and the continuity of the embedding of V into $L^\infty(0,L)$. Here and below, C denotes a generic constant which is independent of β or B_0. Then, we find from Eqs. (5.10), (5.15) and the Lipschitz continuity of p, that

$$
\begin{aligned}
\frac{1}{2}|v_1(t)-v_2(t)|_H^2+B_0\int_0^t\|v_1-v_2\|_V^2\,\mathrm{d}s+A_0\|u_1(t)-u_2(t)\|_V^2 &\leqslant C\int_0^t|u_1-u_2|_H|v_1-v_2|_H\,\mathrm{d}s \\
&+C\int_0^t|\beta_1-\beta_2|_H|v_1-v_2|_H\,\mathrm{d}s.
\end{aligned}
\tag{5.16}
$$

Thus,

$$
|v_1(t)-v_2(t)|_H^2+2B_0\int_0^t\|v_1-v_2\|_V^2\,\mathrm{d}s+A_0\|u_1(t)-u_2(t)\|_V^2\leqslant C\int_0^t|v_1-v_2|_H^2\,\mathrm{d}s+C\int_0^t|\beta_1-\beta_2|_H^2\,\mathrm{d}s,
\tag{5.17}
$$

which implies by Gronwall's inequality, that

$$
|v_1(t)-v_2(t)|_H^2+\|u_1(t)-u_2(t)\|_V^2\leqslant C\int_0^t|\beta_1-\beta_2|_H^2\,\mathrm{d}s,
\tag{5.18}
$$

where C is now allowed to depend on T. From Eq. (5.11) we obtain

$$
\beta(x,t)=\beta_0(x)-\int_0^t\gamma\kappa((u)_+)^2(\beta)_+\,\mathrm{d}s,
$$

and using the inequality (5.14) again along with Eq. (5.18), yields

$$
\begin{aligned}
|\Theta\beta_1(x,t)-\Theta\beta_2(x,t)| &\leqslant C\int_0^t\left|(u_1)_+^2(\Theta\beta_1)_+-(u_2)_+^2(\Theta\beta_2)_+\right|\mathrm{d}s \\
&\leqslant C\int_0^t|\Theta\beta_1(s)(x)-\Theta\beta_2(s)(x)|\,\mathrm{d}s+C\int_0^t\left(\int_0^s|\beta_1-\beta_2|_H^2\,\mathrm{d}r\right)^{1/2}\mathrm{d}s.
\end{aligned}
$$

It follows from Gronwall's inequality and Jensen's inequality that

$$
|\Theta\beta_1(t)-\Theta\beta_2(t)|_H^2\leqslant C\int_0^t|\beta_1(s)-\beta_2(s)|_H^2\,\mathrm{d}s.
$$

Iterating this inequality, we find that for n large enough Θ^n is a contraction mapping on $C(0,T;H)$, and therefore, Θ has a unique fixed point β. From the differential equation satisfied by β, we see that β, $\beta'\in C(0,T;L^\infty(0,L))$. Thus β is the unique solution of our problem. The proof of the theorem is now complete. □

We now consider the dynamic inviscid ($B_0=0$) problem: Find $v\in\mathscr{V}$ and $\beta\in C(0,T;L^\infty(0,L))$ such that

$$
v'+Au+S(\beta,u)+P(u)=f\quad\text{in }\mathscr{V}',
\tag{5.19}
$$

$$\beta' + \gamma\kappa((u)_+)^2(\beta)_+ = 0, \quad \beta' \in C(0,T;L^\infty(0,L)), \tag{5.20}$$

$$\beta(0) = \beta_0, \quad v(0) = v_0, \tag{5.21}$$

$$u(t) = u_0 + \int_0^t v(s)\,\mathrm{d}s. \tag{5.22}$$

We have the following result.

Theorem 5.4. *Assume that* $\beta_0(x) \in (0,1]$, $u_0 \in V$ *and* $v_0 \in H$. *Then, there exists a unique solution of problem* (5.19)–(5.22).

Proof. Letting $B_0 = \delta$, we denote by v_δ, u_δ, β_δ, the solution of the dynamic viscous problem, (5.10)–(5.13) where $\eta = 1$. Thus Eq. (5.10) is of the form

$$v' + B_\delta v + Au + S(\beta,u) + P(u) = f \quad \text{in } \mathscr{V}', \tag{5.23}$$

for

$$\langle B_\delta v, w\rangle \equiv \int_0^L \delta v_{xx} w_{xx}\,\mathrm{d}x. \tag{5.24}$$

Then estimate (5.14) takes the form

$$|v(t)|_H^2 + \delta\int_0^t \|v\|_V^2\,\mathrm{d}s + A_0\|u(t)\|_V^2 \leqslant C, \tag{5.25}$$

where C does not depend on δ. It follows that $\langle B_\delta v_\delta, v_\delta\rangle_{\mathscr{V}} \leqslant C$. Therefore, for $w \in \mathscr{V}$,

$$\langle B_\delta v_\delta, w\rangle_{\mathscr{V}} \leqslant \langle B_\delta v_\delta, v_\delta\rangle_{\mathscr{V}}^{1/2} \langle B_{\delta w}, w\rangle_{\mathscr{V}}^{1/2} \leqslant C\delta^{1/2}\|w\|_{\mathscr{V}}.$$

Thus,

$$B_\delta v_\delta \to 0 \quad \text{strongly in } \mathscr{V}', \tag{5.26}$$

as $\delta \to 0$. From Eq. (5.25) and the boundedness of all the operators, we conclude that there is a subsequence, denoted by $\delta \to 0$, such that in addition to Eq. (5.26),

$$v'_\delta \to v' \quad \text{weak}^* \text{ in } \mathscr{V}', \tag{5.27}$$

$$v_\delta \to v \quad \text{weak}^* \text{ in } L^\infty(0,T;H), \tag{5.28}$$

$$u_\delta \to u \quad \text{weak}^* \text{ in } L^\infty(0,T;V), \tag{5.29}$$

where $u(t) = u_0 + \int_0^t v(s)\,\mathrm{d}s$. By Theorem 5.2, we may also assume that for a subsequence

$$u_\delta \to u \quad \text{strongly in } C(0,T;W), \tag{5.30}$$

where V embeds compactly into W, and W embeds continuously into $C([0,L])$. This strong convergence of u_δ is sufficient to conclude that

$$\beta_\delta \to \beta \quad \text{strongly in } C(0,T;L^\infty(0,L)), \tag{5.31}$$

where β is the solution to Eqs. (5.20) and (5.21). Thus, we also have

$$S(\beta_\delta, u_\delta) \to S(\beta,u) \quad \text{strongly in } L^2(0,T;H), \tag{5.32}$$

$$P(u_\delta) \to P(u) \quad \text{strongly in } L^2(0,T;H). \tag{5.33}$$

Therefore, taking the limit as $\delta \to 0$ in Eqs. (5.10)–(5.13), with B replaced by B_δ and $\eta = 1$, yields a solution for problem (5.19)–(5.22). It only remains to verify uniqueness of the solution.

Suppose v_i and β_i, $i = 1, 2$, are two solutions of Eqs. (5.19)–(5.22). It follows from Eq. (5.16), along with the Lipschitz continuity of p, there exists a constant C, independent of v_i, such that

$$\frac{1}{2}|v_1(t) - v_2(t)|_H^2 + A_0\|u_1(t) - u_2(t)\|_V^2 \leqslant C\int_0^t |u_1 - u_2|_H |v_1 - v_2|_H \,ds + C\int_0^t |\beta_1 - \beta_2|_H |v_1 - v_2|_H \,ds \leqslant C \times \int_0^t |u_1 - u_2|_H^2 \,ds + C\int_0^t |v_1 - v_2|_H^2 \,ds + C\int_0^t |\beta_1 - \beta_2|_H^2 \,ds. \tag{5.34}$$

Now, from Eqs. (5.20) and (5.14), which is an estimate of $\|u(t)\|_{L^\infty(0,L)}$ due to the continuity of the embedding of V into $L^\infty(0,L)$, we obtain

$$\frac{1}{2}|\beta_1(t) - \beta_2(t)|_H^2 \leqslant C\int_0^t |u_1 - u_2|_H |\beta_1 - \beta_2|_H \,ds \leqslant C\int_0^t |u_1 - u_2|_H^2 \,ds + C\int_0^t |\beta_1 - \beta_2|_H^2 \,ds. \tag{5.35}$$

Adding Eqs. (5.34) and (5.35) and using Gronwall's inequality yields $v_1 = v_2$, $u_1 = u_2$, and $\beta_1 = \beta_2$. □

We consider next the manner in which solutions of the quasistatic problems with normal compliance approach the solution of the quasistatic problem of Section 3, in which no penetration is allowed. Writing the problem of Section 3 in terms of operators, as in this section, we obtain the following problem,

$$u(t) \in K, \tag{5.36}$$

$$\langle Au, u - w\rangle + \langle S(\beta, u), u - w\rangle \leqslant (f(t), u - w)_H, \quad w \in K \tag{5.37}$$

$$\beta' + \gamma\kappa u^2 \beta_+ = 0, \quad \beta, \beta' \in C(0,T;L^\infty(0,L)), \tag{5.38}$$

$$\beta(0)(x) = \beta_0(x) \in (0,1] \quad \text{a.e. on } (0,L). \tag{5.39}$$

We denote by u_ε, β_ε the solution of the quasistatic normal compliance problem in which the normal compliance is penalized by multiplying it with $1/\varepsilon$, as $\varepsilon \to 0$. Thus,

$$Au + S(\beta, u) + \frac{1}{\varepsilon}P(u) = f, \tag{5.40}$$

$$\beta' + \gamma\kappa((u)_+)^2(\beta)_+ = 0, \quad \beta, \beta' \in C(0,T;L^\infty(0,L)), \tag{5.41}$$

$$\beta(0)(x) = \beta_0(x) \in (0,1] \quad \text{a.e. on } (0,L). \tag{5.42}$$

We let W be any space for which the embedding of V into W is compact. We have the following convergence result, which guarantees that as the support becomes more rigid the solution gets closer to that of the problem with a rigid obstacle. The main idea of the proof is in establishing the equicontinuity of the sequence of the solutions $\{u_\varepsilon\}$ for the problems with normal compliance, and using the uniqueness of the limit.

Theorem 5.5. *The solutions u_ε of problems* (5.40)–(5.42) *converge strongly in $C(0,T;W)$ and weak* in $L^\infty(0,T;V)$ to the solution of problem* (5.36)–(5.39).

Proof. Using the fact that $0 \in K$, we multiply Eq. (5.40) by u_ε and let $w = 0$ in Eq. (5.37) and obtain the estimates

$$\|u\|_V, \ \|u_\varepsilon\|_V \leqslant C, \tag{5.43}$$

for a constant C that is independent of ε.

To establish the equicontinuity of $\{u_\varepsilon\}$ we need to consider the continuity of the map $t \to u_\varepsilon(t)$. We let $u = u_\varepsilon$ be the solution of Eq. (5.40), and for the sake of simplicity we do not indicate explicitly the dependence on $x \in [0, L]$. We have, for s, $t \in [0, T]$ and $t < s$, that

$$Au(s) - Au(t) + S(\beta(s), u(s)) - S(\beta(t), u(t)) + \frac{1}{\varepsilon}(P(u(s)) - P(u(t))) = f(s) - f(t).$$

Multiplying this expression by $u(s) - u(t)$, integrating and using the estimate (5.43) along with the definition of the map $S(\beta, u)$, we find

$$\|u(s) - u(t)\|_V^2 - \kappa C|\beta(s) - \beta(t)|_H |u(s) - u(t)|_H \leqslant |f(s) - f(t)|_H |u(s) - u(t)|_H,$$

where here and below, C is a generic constant which is independent of ε or $x \in [0, L]$. Therefore,

$$\|u(s) - u(t)\|_V^2 \leqslant C\Big(|f(s) - f(t)|_H^2 + |\beta(s) - \beta(t)|_H^2\Big). \tag{5.44}$$

Now, we consider the differential equation satisfied by β. From Eq. (5.41) we have

$$\beta(s) - \beta(t) = \int_t^s (-\gamma\kappa u_+^2 \beta_+)\,\mathrm{d}r,$$

and so

$$|\beta(s) - \beta(t)| \leqslant C|s - t|.$$

Therefore, Eq. (5.44) implies

$$\|u(s) - u(t)\|_V^2 \leqslant C\Big(|f(s) - f(t)|_H^2 + |s - t|^2\Big),$$

which shows that the set $\{u_\varepsilon\}$ is an equicontinuous subset of $C(0, T; V)$ since by assumption (3.2) $f \in C(0, T; H)$. In addition, Eq. (5.43) implies that the set $\{u_\varepsilon\}$ is uniformly bounded. It follows from the Ascoli–Arzela theorem that $\{u_\varepsilon\}$ is precompact in $C(0, T; W)$. By standard arguments there exists a subsequence $\varepsilon_k \to 0$, such that $u_{\varepsilon_k} \to u$ weak* in $L^\infty(0, T; V)$ and strongly in $C(0, T; W)$, where u is the solution of problem (5.36)–(5.39). Since the solution of problem (5.36)–(5.39) is unique, it follows that it is not necessary to take a subsequence, and the whole sequence of solutions of the penalized problems (5.40)–(5.42) converges weak* in $L^\infty(0, T; V)$ and strongly in $C(0, T; W)$, whenever W is a space as described above. □

6. Semi-discrete approximation of the quasistatic problem

We now turn to numerical approximations of the quasistatic elastic problem with normal compliance studied in the previous section. We establish the convergence of the scheme and obtain an error estimate. We analyze first a spatially semi-discrete scheme, while a fully discrete scheme is the subject of the next section. The dynamic problem will be investigated in the future.

We note that the numerical analysis of the quasistatic problem with the Signorini non-penetration condition (2.6), studied in Sections 2 and 3, is hampered by the following difficulty. The solution of the problem lies in $H^2(0, L)$ and we require, in addition, that $0 \leqslant u$, but a general function satisfying this inequality does not lie in $H^2(0, L)$, since it may be only Lipschitz continuous. This means that a general approximation of the problem will either violate the Signorini condition or will not be in V, so in either case the necessary estimates cannot be obtained.

The problem is to find (u, β) such that

$$a(u(t), v) + (\kappa\beta^2(t)u(t), v)_H = (f(t) + p(u(t)), v)_H \quad \text{for all } v \in V, \tag{6.1}$$

$$\beta' + \gamma\kappa(u_+)^2(\beta)_+ = 0 \quad \text{in } \Omega_T, \tag{6.2}$$

$$\beta(0) = \beta_0 \quad \text{a.e. on } (0, L). \tag{6.3}$$

We introduce a partition of the spatial domain $[0, L]$: $0 = x_0 < x_1 < \cdots < x_M = L$. Denote $I_i = [x_{i-1}, x_i]$ and $h_i = x_i - x_{i-1}$ for $i = 1, \ldots, M$, and $h = \max_{1 \leqslant i \leqslant M} h_i$ the meshsize. We define the finite element spaces

$$V^h = \{v^h \in V | v^h|_{I_i} \text{ is cubic}, \ 1 \leqslant i \leqslant M\}, \tag{6.4}$$

$$Q^h = \{q^h \in L^\infty(0, L) | q^h|I_i \text{ is constant}, \ 1 \leqslant i \leqslant M\}. \tag{6.5}$$

Thus, V^h consists of piecewise cubics, and Q^h of piecewise constant functions. We observe that an equivalent definition of the space V^h is

$$V^h = \{v^h \in C^1([0, L]) | v^h(0) = v_x^h(0) = 0, v^h|_{I_i} \text{ is cubic}, \ 1 \leqslant i \leqslant M\}.$$

We define a piecewise averaging operator $\mathscr{P}^h : L^1(0, L) \to Q^h$ by

$$\mathscr{P}^h u|_{I_i} = \frac{1}{|I_i|} \int_{I_i} u \, \mathrm{d}x, \quad 1 \leqslant i \leqslant M, \quad u \in L^1(0, L). \tag{6.6}$$

Obviously, the operator $\mathscr{P}^h$ has the property

$$\|\mathscr{P}^h u\|_W \leqslant \|u\|_W \quad \forall u \in W, \tag{6.7}$$

where W is any one of $L^\infty(0, L)$, $L^\infty(I_i)$, H or $L^2(I_i)$. Then, a spatially semi-discrete scheme of the quasistatic problem with normal compliance is:

Problem $\mathrm{P}^h_{\mathrm{NC}}$. Find $u^h : [0, T] \to V^h$ and $\beta^h : [0, T] \to Q^h$ such that for all $t \in [0, T]$,

$$a(u^h(t), v^h) + (\kappa(\beta^h)^2 u^h(t), v^h)_H = \big(f(t) + p(u^h(t)), v^h\big)_H \quad \forall v^h \in V^h, \tag{6.8}$$

$$(\beta^h)' + \gamma\mathscr{P}^h[\kappa(u_+^h)^2](\beta^h)_+ = 0 \quad \text{in } \Omega_T, \tag{6.9}$$

$$\beta^h(0) = \beta_0^h \quad \text{on } (0, L), \tag{6.10}$$

where $\beta_0^h \in Q^h$ is an approximation of β_0.

Using the proof technique of Section 5, it is not difficult to show that the semi-discrete problem has a unique solution $(u^h, \beta^h) \in C(0, T; V^h) \times C^1(0, T; Q^h)$ and for some constant $C > 0$, we have

$$\|u^h\|_{C(0,T;V)} \leqslant C \quad \forall h > 0. \tag{6.11}$$

Our purpose here is to analyze the convergence, and derive error estimates for the solution of $\mathrm{P}^h_{\mathrm{NC}}$. Since the function p is decreasing and globally Lipschitz, we have

$$(p(w_1) - p(w_2))(w_1 - w_2) \leqslant 0, \tag{6.12}$$

$$|p(w_1) - p(w_2)| \leqslant L_p |w_1 - w_2|, \tag{6.13}$$

for some constant $L_p > 0$ and for all w_1, $w_2 \in \mathrm{R}$.

Let us derive now error relations. From Eqs. (6.1) and (6.8) we obtain, for $t \in [0,T]$. The first error relation:

$$a(u(t)-u^h(t),v^h) = (p(u(t))-p(u^h(t)),v^h)_H - (\kappa(\beta(t)^2u(t)-\beta^h(t)^2u^h(t)),v^h)_H \quad \forall v^h \in V^h. \tag{6.14}$$

Integrating Eq. (6.2) from 0 to t and using the initial value (6.3), yields

$$\beta(t) = \beta_0 - \gamma\int_0^t \kappa u_+(s)^2\beta_+(s)\,\mathrm{d}s. \tag{6.15}$$

Similarly, Eqs. (6.9) and (6.10) imply

$$\beta^h(t) = \beta_0^h - \gamma\int_0^t \mathscr{P}^h[\kappa u_+^h(s)^2]\beta_+^h(s)\,\mathrm{d}s. \tag{6.16}$$

Subtracting Eq. (6.16) from Eq. (6.15), we get the second error relation:

$$\beta(t)-\beta^h(t) = \beta_0-\beta_0^h - \gamma\int_0^t \left(\kappa u_+(s)^2\beta_+(s) - \mathscr{P}^h[\kappa u_+^h(s)^2]\beta_+^h(s)\right)\mathrm{d}s. \tag{6.17}$$

For $v^h \in C(0,T;V^h)$, we write

$$a(u(t)-u^h(t),u(t)-u^h(t)) = a(u(t)-u^h(t),u(t)-v^h(t)) + a(u(t)-u^h(t),v^h(t)-u^h(t)).$$

Using Eq. (6.14), we have

$$\begin{aligned} a(u(t)-u^h(t),v^h(t)-u^h(t)) = {} & (p(u(t))-p(u^h(t)),v^h(t)-u^h(t))_H - (\kappa(\beta(t)^2u(t) \\ & - (\beta^h(t))^2u^h(t)),v^h(t)-u^h(t))_H. \end{aligned} \tag{6.18}$$

Now, by the properties (6.12) and (6.13),

$$\begin{aligned} (p(u(t))-p(u^h(t)),v^h(t)-u^h(t))_H &\leqslant (p(u(t))-p(v^h(t)),v^h(t)-u^h(t))_H \\ &\leqslant C\|u(t)-v^h(t)\|_H\|v^h(t)-u^h(t)\|_H \\ &\leqslant C\|u(t)-v^h(t)\|_H(\|u(t)-v^h(t)\|_H + \|u(t)-u^h(t)\|_H) \\ &\leqslant C\|u(t)-v^h(t)\|_H^2 + \delta_1\|u(t)-u^h(t)\|_H^2, \end{aligned}$$

with a small number $\delta_1 > 0$ to be chosen later. Also,

$$\begin{aligned} &-(\kappa(\beta(t)^2u(t)-(\beta^h(t))^2u^h(t)),v^h(t)-u^h(t)) \leqslant -(\kappa(\beta(t)^2u(t)-(\beta^h(t))^2v^h(t)),v^h(t)-u^h(t)) \\ &\quad = -(\kappa(\beta(t)^2-(\beta^h(t))^2)u(t)+\kappa(\beta^h(t))^2(u(t)-v^h(t)),v^h(t)-u^h(t)) \leqslant C\|u(t)-v^h(t)\|_H^2 + C\|\beta(t) \\ &\qquad - \beta^h(t)\|_H^2 + \delta_2\|u(t)-u^h(t)\|_H^2, \end{aligned}$$

with a small number $\delta_2 > 0$ to be chosen below. From Eq. (6.18) and the V-ellipticity of $a(\cdot,\cdot)$ we have

$$\|u(t)-u^h(t)\|_V^2 \leqslant C(\|u(t)-v^h(t)\|_V^2 + \|\beta(t)-\beta^h(t)\|_H^2) + C(\delta_1+\delta_2)\|u(t)-u^h(t)\|_H^2.$$

Choosing δ_1 and δ_2 sufficiently small so that $C(\delta_1+\delta_2) < 1$, we obtain

$$\|u(t)-u^h(t)\|_V \leqslant C(\|u(t)-v^h(t)\|_V + \|\beta(t)-\beta^h(t)\|_H). \tag{6.19}$$

We now derive an estimate for $\beta(t)-\beta^h(t)$ based on Eq. (6.17). Write

$$\begin{aligned} \kappa u_+(s)^2\beta_+(s) - \mathscr{P}^h[\kappa u_+^h(s)^2]\beta_+^h(s) = {} & \mathscr{P}^h[\kappa u_+^h(s)^2](\beta_+(s)-\beta_+^h(s)) + \beta_+(s)\mathscr{P}^h[\kappa(u_+(s)^2-(u_+^h(s))^2)] \\ & + \beta_+(s)(\kappa u_+(s)^2 - \mathscr{P}^h[\kappa u_+(s)^2]). \end{aligned}$$

Then Eqs. (6.7) and (6.11) imply that $|\mathscr{P}^h[\kappa u_+^h(s)^2]| \leqslant C$ for $h > 0$, and for all $s \in [0, T]$. Hence,

$$|\kappa u_+(s)^2 \beta_+(s) - \mathscr{P}^h[\kappa u_+^h(s)^2]\beta_+^h(s)| \leqslant C|\beta_+(s) - \beta_+^h(s)| + C|\mathscr{P}^h[\kappa(u_+(s)^2 - u_+^h(s)^2)]| + C|\kappa u_+(s)^2 - \mathscr{P}^h[\kappa u_+(s)^2]|. \tag{6.20}$$

Then, from Eq. (6.17), we get

$$\|\beta(t) - \beta^h(t)\|_H \leqslant \|\beta_0 - \beta_0^h\|_H + C\int_0^t (\|\beta(s) - \beta^h(s)\|_H + \|u(s)^2 - u^h(s)^2\|_H + \|\kappa u_+(s)^2 - \mathscr{P}^h[\kappa u_+(s)^2]\|_H)\,\mathrm{d}s.$$

and then

$$\|\beta(t) - \beta^h(t)\|_H \leqslant \|\beta_0 - \beta_0^h\|_H + C\int_0^t (\|\beta(s) - \beta^h(s)\|_H + \|u(s) - u^h(s)\|_H + \|\kappa u_+(s)^2 - \mathscr{P}^h[\kappa u_+(s)^2]\|_H)\,\mathrm{d}s. \tag{6.21}$$

Combining Eqs. (6.19) and (6.21), we find

$$\|u(t) - u^h(t)\|_V + \|\beta(t) - \beta^h(t)\|_H \leqslant C(\|\beta_0 - \beta_0^h\|_H + \|u(t) - v^h(t)\|_V + \|\kappa u_+^2 - \mathscr{P}^h(\kappa u_+^2)\|_{L^1(0,T;H)}) + C\int_0^t \left(\|\beta(s) - \beta^h(s)\|_H + \|u(s) - u^h(s)\|_V\right)\mathrm{d}s. \tag{6.22}$$

Applying the Gronwall inequality, we obtain

$$\|u - u^h\|_{L^\infty(0,T;V)} + \|\beta - \beta^h\|_{L^\infty(0,T;H)} \leqslant C(\|\beta_0 - \beta_0^h\|_H + \|u - v^h\|_{L^\infty(0,T;V)} + \|\kappa u_+^2 - \mathscr{P}^h(\kappa u_+^2)\|_{L^1(0,T;H)}), \tag{6.23}$$

for all $v^h \in C(0, T; V^h)$.

Since $u \in C(0, T; V)$, we have $u_+^2 \in C(0, T; H^1(0, L))$. Under the additional assumption $\kappa \in H^1(0, L)$, we have (cf. Ciarlet, 1978; Quarteroni and Valli, 1994)

$$\|\kappa u_+^2 - \mathscr{P}^h(\kappa u_+^2)\|_{L^1(0,T;H)} \leqslant Ch,$$

where C is proportional to $\|u_+^2\|_{C(0,T;H^1(0,L))}$.

Since $v^h \in C(0, T; V^h)$ is arbitrary in Eq. (6.23), by following the arguments in Han and Reddy (1999, Chapter 11) we obtain the next theorem.

Theorem 6.1. *Assume the initial value β_0^h is chosen so that*

$$\|\beta_0 - \beta_0^h\|_H \to 0 \quad \text{as } h \to 0. \tag{6.24}$$

Then the semi-discrete approximation method converges,

$$\|u - u^h\|_{L^\infty(0,T;V)} + \|\beta - \beta^h\|_{L^\infty(0,T;H)} \to 0 \quad \text{as } h \to 0.$$

If, in addition, we assume $\kappa \in H^1(0, L)$ and $u \in L^\infty(0, T; H^3(0, L))$ and

$$\|\beta_0 - \beta_0^h\|_H \leqslant Ch, \tag{6.25}$$

then we have the error estimate

$$\|u - u^h\|_{L^\infty(0,T;V)} + \|\beta - \beta^h\|_{L^\infty(0,T;H)} \leqslant Ch.$$

We note that if we take $\beta_0^h = \mathscr{P}^h\beta_0$, then the condition (6.24) is guaranteed for $\beta_0 \in H$ while the condition (6.25) is guaranteed for $\beta_0 \in H^1(0, L)$.

7. Fully discrete approximations

To develop a fully discrete scheme, in addition to the partition of the spatial domain $[0, L]$ introduced in the previous section, we need a partition of the time interval $[0, T]$: $0 = t_0 < t_1 < \cdots < t_N = T$. We denote the step-size $k_n = t_n - t_{n-1}$ for $n = 1, \ldots, N$. We allow non-uniform partition of the time interval, and denote by $k = \max_n k_n$ the maximal step-size. For a continuous function $w(t)$, we use the notation $w_n = w(t_n)$.

A fully discrete scheme for the quasistatic problem with normal compliance is:

Problem $\mathrm{P}^{\mathrm{hk}}_{\mathrm{NC}}$. Find $\{u_n^{hk}, \beta_n^{hk}\}_{n=0}^N \subset V^h \times Q^h$ such that for $n = 1, \ldots, N$,

$$a(u_n^{hk}, v^h) + (\kappa(\beta_n^{hk})^2 u_n^{hk}, v^h) = (f_n + p(u_n^{hk}), v^h) \quad \forall v^h \in V^h, \tag{7.1}$$

$$\beta_n^{hk} - \beta_{n-1}^{hk} + \gamma k_n \mathscr{P}^h[\kappa(u_{n-1}^{hk})_+^2](\beta_{n-1}^{hk})_+ = 0, \tag{7.2}$$

and

$$u_0^{hk} = u_0^h, \beta_0^{hk} = \beta_0^h \quad \text{on } (0, L), \tag{7.3}$$

where $\beta_0^h \in Q^h$ is an approximation of β_0 and $u_0^h \in V^h$.

Here, $u_0^h \in V^h$ is an *artificial* initial value required by the fully discrete scheme. This value is needed in the explicit discretization (7.2) with $n = 1$. Such an artificial initial value can be avoided if we replace Eq. (7.2) by the implicit discretization

$$\beta_n^{hk} + \gamma k_n \mathscr{P}^h[\kappa(u_n^{hk})_+^2](\beta_{n-1}^{hk})_+ = \beta_{n-1}^{hk}.$$

However, then this scheme is much more difficult to analyze or use. Although we use the symbol u_0^h, it does not represent an approximation of $u(0)$. Moreover, as the error analysis below suggests, we have the freedom to use any value in a bounded set for u_0^h (e.g., $u_0^h = 0$) without decreasing the convergence order of the method.

The fully discrete solution exits and is unique, and for some constant c_1, $C_2 > 0$,

$$-c_1 k \leqslant \beta_n^{hk} \leqslant 1, \|u_n^{hk}\|_V \leqslant C_2, \quad n = 0, 1, \ldots, N, \ \forall h, k > 0.$$

From Eqs. (6.1) and (7.1) we have the first error relation,

$$a(u_n - u_n^{hk}, v^h) = (p(u_n) - p(u_n^{hk}), v^h) - (\kappa(\beta_n^2 u_n - (\beta_n^{hk})^2 u_n^{hk}), v^h) \quad \forall v^h \in V^h. \tag{7.4}$$

From Eqs. (7.2) and (7.3), we have

$$\beta_n^{hk} = \beta_0^h - \gamma \sum_{j=1}^n k_j \mathscr{P}^h[\kappa(u_{j-1}^{hk})_+^2](\beta_{j-1}^{hk})_+. \tag{7.5}$$

Combining Eq. (6.15) at $t = t_n$ and Eq. (7.5) we obtain the second error relation,

$$\beta_n - \beta_n^{hk} = \beta_0 - \beta_0^h - \gamma \sum_{j=1}^n \int_{t_{j-1}}^{t_j} [\kappa u_+(s)^2 \beta_+(s) - \mathscr{P}^h[\kappa(u_{j-1}^{hk})_+^2](\beta_{j-1}^{hk})_+] \, \mathrm{d}s. \tag{7.6}$$

Similarly to the arguments leading to Eq. (6.19), we conclude from Eq. (7.4) that

$$\|u_n - u_n^{hk}\|_V \leqslant C(\|u_n - v_n^h\|_V + \|\beta_n - \beta_n^{hk}\|_H), \tag{7.7}$$

for any $v_n^h \in V^h$. Here and below, C is a positive constant independent of k or h. Writing

$$\kappa u_+(s)^2\beta_+(s) - \mathscr{P}^h[\kappa(u^{hk}_{j-1})^2_+](\beta^{hk}_{j-1})_+ = \kappa(u_{j-1})^2_+[\beta_+(s) - (\beta_{j-1})_+] + \kappa[u_+(s) + (u_{j-1})_+]\beta_+(s)[u_+(s) - (u_{j-1})_+] + \kappa(u_{j-1})^2_+(\beta_{j-1})_+ - \mathscr{P}^h[\kappa(u^{hk}_{j-1})^2_+](\beta^{hk}_{j-1})_+,$$

and using Eq. (6.20), we obtain the following inequality from Eq. (7.6):

$$\|\beta_n - \beta_n^{hk}\|_H \leqslant \|\beta_0 - \beta_0^h\|_H + C\sum_{j=1}^n \int_{t_{j-1}}^{t_j} (\|\beta_+(s) - (\beta_{j-1})_+\|_H + \|u_+(s) - (u_{j-1})_+\|_H)\,\mathrm{d}s + C\sum_{j=1}^n k_j(\|\beta_{j-1} - \beta_{j-1}^{hk}\|_H + \|u_{j-1} - u_{j-1}^{hk}\|_H + \|\kappa(u_{j-1})_+^2 - \mathscr{P}^h[\kappa(u_{j-1})_+^2]\|_H). \tag{7.8}$$

Combining Eqs. (7.7) and (7.8) leads to

$$||u_n - u_n^{hk}||_V + ||\beta_n - \beta_n^{hk}||_H \leqslant C||\beta_0 - \beta_0^h||_H + C||u_n - v_n^h||_V + C\sum_{j=1}^n \int_{t_j-1}^{t_j} (||\beta_+(s) - (\beta_{j-1})_+||_H + ||u_+(s) - (u_{j-1})_+||_H)\,\mathrm{d}s + C\sum_{j=1}^n k_j||\kappa(u_{j-1})_+^2 - \mathscr{P}^h[\kappa(u_{j-1})_+^2]||_H + C\sum_{j=1}^n k_j\Big(||\beta_{j-1} - \beta_{j-1}^{hk}||_H + ||u_{j-1} - u_{j-1}^{hk}||_H\Big).$$

Since v_n^h is arbitrary in V^h, applying a discrete Gronwall inequality (cf. Han and Sofonea (2000), Lemma 4.1) we have

$$\max_{1\leqslant n\leqslant N}\big[||u_n - u_n^{hk}||_V + ||\beta_n - \beta_n^{hk}||_H\big] \leqslant C||\beta_0 - \beta_0^h||_H + Ck||u(0) - u_0^h||_H + C\sum_{j=1}^N \int_{t_{j-1}}^{t_j} \big(||\beta_+(s) - (\beta_{j-1})_+||_H + ||u_+(s) - (u_{j-1})_+||_H\big)\,\mathrm{d}s + C\sum_{j=1}^N k_j||\kappa(u_{j-1})_+^2 - \mathscr{P}^h[\kappa(u_{j-1})_+^2]||_H + C\max_{1\leqslant n\leqslant N}\inf_{v_n^h\in V^h}||u_n - v_n^h||_V. \tag{7.9}$$

Inequality (7.9) is the basis for convergence analysis and error estimation. Since $u_+,\ \beta_+ \in C(0,T;H)$, then $u_+(s)$ and $\beta_+(s)$ are uniformly continuous on $[0,T]$ in the norm $||\cdot||_H$. Hence,

$$\sum_{j=1}^N \int_{t_{j-1}}^{t_j} \big(||\beta_+(s) - (\beta_{j-1})_+||_H + ||u_+(s) - (u_{j-1})_+||_H\big)\,\mathrm{d}s \to 0 \quad \text{as } k \to 0.$$

Under the assumption $(u_+)', (\beta_+)' \in W^{1,\infty}(0,T;H)$, we have

$$\sum_{j=1}^N \int_{t_{j-1}}^{t_j} \big(||\beta_+(s) - (\beta_{j-1})_+||_H + ||u_+(s) - (u_{j-1})_+||_H\big)\,\mathrm{d}s \leqslant Ck.$$

Summarizing our findings, we have the following result concerning convergence and convergence order of the fully discrete scheme.

Theorem 7.1. *Assume $||u_0^h||_H$ is uniformly bounded with respect to h and the initial value β_0^h is chosen so that*

$$||\beta_0 - \beta_0^h||_H \to 0 \quad \text{as } h \to 0. \tag{7.10}$$

Then, the fully discrete method converges, thus

$$\max_{1\leqslant n\leqslant N}\big[||u_n - u_n^{hk}||_V + ||\beta_n - \beta_n^{hk}||_H\big] \to 0 \quad \text{as } h,k \to 0.$$

If, in addition, we assume $\kappa \in H^1(0,L)$, $u \in L^\infty(0,T;H^3(0,L))$, $(u_+)'$, $(\beta_+)' \in W^{1,\infty}(0,T;H)$, $||u_0^h||_H$ *is uniformly bounded with respect to h and*

$$||\beta_0 - \beta_0^h||_H \leqslant Ch, \tag{7.11}$$

then, we have the error estimate

$$\max_{1 \leqslant n \leqslant N} \left[||u_n - u_n^{hk}||_V + ||\beta_n - \beta_n^{hk}||_H \right] \leqslant C(h+k).$$

Again we note that if we take $\beta_0^h = \mathscr{P}^h \beta_0$, then conditions (7.10) and (7.11) are easily satisfied.

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